ON THE VARIATION OF THE ARBITRARY AND GIVEN CONSTANTS

IN DYNAMICAL EQUATIONS*

BY

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Introduction.

The method of the variation of arbitrary constants is a well-known device for
obtaining the solution of a set of differential equations when we know that of
another set which differs from the former in a suitable manner.   Especially is
it useful in most of the problems of celestial mechanics where methods of approxi-
mation have to be adopted.   The first approximation once obtained, it gives
the second approximation with ease and certainty, and the same idea can be fol-
lowed out for the higher approximations if further accuracy is desired.

In the class of problems under consideration here, the equations of motion
can be put into the form

$$\frac{d^2 x_i}{dt^2} = \frac{\partial F}{\partial x_i} \qquad\qquad (i = 1, 2, \cdots n),$$

or into the form, known as canonical,

$$\frac{d x_i}{dt} = \frac{\partial F}{\partial X_i}, \qquad \frac{d X_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad\qquad (i = 1, 2, \cdots n),$$

where F is a function either of the x_i or of the x_i, X_i, respectively, and may
or may not contain the time explicitly.   Suppose that the problem has been
solved so that the x_i, X_i are expressed in terms of $2n$ arbitrary constants and
the time, and that it be required to find the solution of another problem in which
$F + R$ replaces F.   The new problem can be stated by supposing the arbitrary
constants in the first problem to be variable and finding the differential equations
satisfied by them.

The function F will usually contain, besides x_i, t, a number of given con-
stants which will naturally be present in the solution, and it may happen that
some of these constants are really variables which, in order to get a first approxi-
mation, we have treated as constants.   It is, of course, not possible in general

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to solve the equations as though they were constant and then substitute their variable values in the solution.   If we follow the method mentioned we may ask what variable values are to be given to the arbitraries in order that the variable values of the given constants may be substituted in the solution.   Thus instead of adding R to F the new problem is that obtained by replacing some of the constants present in F by functions of the time; it is chiefly this problem that I propose to consider in the following pages.

In order to do so it is necessary to consider in some detail the relations satisfied by the derivatives of the dependent variables, not only with respect to the arbitraries but also with respect to the given constants.   In the course of this investigation, generalizations of certain theorems which I gave in 1897 * (which included generalizations of ADAMS' theorems connecting the mean motions of the perigee and node with the constant term in the expression for the moon's parallax) are obtained and some new relations between the various constants are added (§§ 1–4).   In § 5 the equations for the variable values of the arbitraries corresponding to variable values for the given constants are obtained.   It there appears that NEWCOMB's theorem (see art. 24) is a particular case of a general theorem which applies to all equations of dynamics satisfying the conditions laid down.   In § 6 applications of the results obtained are made to the lunar theory and in particular to the effects which variable values for the given constants produce on the motions of the arguments.

§ 1. THE EQUATIONS OF VARIATIONS.

1. Let (x, y, z) be the coördinates, referred to fixed axes, of a particle moving under a force-function F.   The equations of motion are

$$(1) \qquad \frac{d^2 x}{dt^2} = \frac{\partial F}{\partial x}, \qquad \frac{d^2 y}{dt^2} = \frac{\partial F}{\partial y}, \qquad \frac{d^2 z}{dt^2} = \frac{\partial F}{\partial z}.$$

The solution of these equations involves a certain number of constants: of these, six are arbitrary and the others are known and present in F.

Denote by f, g any two of these constants and differentiate (1) with respect to f, g successively.   We find

$$\frac{d^2}{dt^2} \frac{dx}{df} = \frac{\partial^2 F}{\partial x^2} \frac{dx}{df} + \frac{\partial^2 F}{\partial x \partial y} \frac{dy}{df} + \frac{\partial^2 F}{\partial x \partial z} \frac{dz}{df} + \frac{d'}{df} \frac{\partial F}{\partial x},$$

$$\frac{d^2}{dt^2} \frac{dx}{dg} = \frac{\partial^2 F}{\partial x^2} \frac{dx}{dg} + \frac{\partial^2 F}{\partial x \partial y} \frac{dy}{dg} + \frac{\partial^2 F}{\partial x \partial z} \frac{dz}{dg} + \frac{d'}{dg} \frac{\partial F}{\partial x},$$

with similar equations for the other coördinates; the operators d'/df, d'/dg

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* *On certain properties of the mean motions*, etc., Proceedings of the London Mathematical Society, vol. 28 (1896), pp. 143-155.

denote derivatives with respect to f, g only in so far as they occur explicitly in F.

Multiply these equations by dx/dg, $-\,dx/df$ respectively, and add the corresponding results for the other two coördinates. Denoting by S a sum of similar expressions with respect to the three coördinates, we obtain an equation which may be written,

$$(2) \qquad \frac{d}{dt}(f,g) = S\left(\frac{dx}{dg}\frac{d'}{df}\frac{\partial F}{\partial x} - \frac{dx}{df}\frac{d'}{dg}\frac{\partial F}{\partial x}\right) = \frac{d}{dg}\frac{d'F}{df} - \frac{d}{df}\frac{d'F}{dg},$$

where

$$(3) \qquad (f,g) = S\left(\frac{dx}{dg}\frac{d}{dt}\frac{dx}{df} - \frac{dx}{df}\frac{d}{dt}\frac{dx}{dg}\right) = -\,(g,f).$$

If h be another of the constants we have, by direct differentiation,

$$(4) \qquad \frac{d}{df}(g,h) + \frac{d}{dg}(h,f) + \frac{d}{dh}(f,g) = 0.$$

2. The constants will now be separated into two classes. Suppose that the solution is such that F and each coördinate is expressible by means of sums of cosines or sines of sums of multiples of the angles

$$b_i t + \beta_i = w_i,$$

where b_i and the coefficients of the periodic terms depend only on one set of constants α_i, the other set being β_i, and the α_i, β_i being all independent of one another; these will be respectively called the α-set and the β-set. It is supposed that F is given in terms of x, y, z and the known constants, involving the latter in the form just given. It is also supposed that the expressions for F and for its derivatives contain no term independent of x, y, z. Also the constant of length is to be such a function of the arbitrary constants of the solution that x, y, z, the kinetic energy T, and F contain no term independent of the arbitraries. In general, we shall avoid choosing the constant of length as an arbitrary so that certain functions which are only defined by their derivatives with respect to the arbitraries need no added constant independent of them.

3. Let α, α' be two constants of the α-set, and β, β' two of the β-set corresponding to angles w, w'. Denote by $\partial/\partial\alpha$ derivatives with respect to α only in so far as α is contained in the coefficients, so that d/df refers to derivatives when α_i, β_i are considered independent and $\partial/\partial f$ to derivatives when α_i, w_i are considered independent; evidently $\partial/\partial w_i = \partial/\partial\beta_i = d/dw_i = d/d\beta_i$.

Denote by $\dot{x}$, $\dot{y}$, $\dot{z}$ derivatives with respect to t. Since $\dot{x} = \Sigma_i b_i \, dx/d\beta_i$, we have

$$\frac{dx}{d\alpha} = \frac{\partial x}{\partial \alpha} + t\Sigma_i \frac{\partial x}{\partial w_i} \frac{db_i}{d\alpha},$$

$$\frac{d}{dt}\left(\frac{dx}{d\alpha}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial \alpha}.\right) + \Sigma_i \frac{\partial x}{\partial w_i} \frac{db_i}{d\alpha} + t\Sigma_i \frac{\partial \dot{x}}{\partial w_i} \frac{db_i}{d\alpha}$$

$$= \frac{\partial \dot{x}}{\partial \alpha} + t\Sigma_i \frac{\partial \dot{x}}{\partial w_i} \frac{db_i}{d\alpha},$$

in which it is to be noted that $\partial/\partial\alpha$, d/dt are not commutative although $\partial/\partial w_i$, d/dt are so.

Substituting these results in (3) and putting

$$[f, g] = S\left(\frac{\partial \dot{x}}{\partial f} \frac{\partial x}{\partial g} - \frac{\partial x}{\partial f} \frac{\partial \dot{x}}{\partial g}\right) = -[g, f],$$

we obtain

$$(\alpha, \alpha') = [\alpha, \alpha'] + t\Sigma_i \left\{ \frac{db_i}{d\alpha} [w_i, \alpha'] - \frac{db_i}{d\alpha'} [w_i, \alpha] \right\}$$

$$+ t^2 \Sigma_i \Sigma_j \frac{db_i}{d\alpha} \frac{db_j}{d\alpha'} [w_i, \dot{w}_j],$$

(5)
$$(\alpha, \beta) = [\alpha, w] + t\Sigma_i \frac{db_i}{d\alpha} [w_i, w],$$

$$(\beta, \beta') = [w, w'].$$

Also

(6)
$$\frac{\partial}{\partial f}[g, h] + \frac{\partial}{\partial g}[h, f] + \frac{\partial}{\partial h}[f, g] = 0.$$

The quantities (f, g) must be divided into several classes, each of which requires separate consideration. The constants f, g may both belong to the α-set, or to the β-set, or one to the α-set and the other to the β-set. Again, they may both be arbitrary, or both be known, or one arbitrary and one known. I shall only consider two known constants—one belonging to each set—so that there will be eight different kinds of brackets to consider, three of them being independent of the known constants. The extension to any number of known constants is made without difficulty.

§ 2. APPLICATION TO THE ARBITRARY CONSTANTS.

4. When the constants considered in § 1 are the arbitraries of the solution, the right hand member of (2) is zero since F does not contain them explicitly; all the (f, g) are therefore constant. But $[\alpha, \alpha']$, $[\beta, \beta']$ consist only of

sums of sines owing to the forms assumed for the coördinates.   Hence by § 1, art. 3,

(7) $(\alpha, \alpha') = [\alpha, \alpha'] = (\beta, \beta') = [w, w'] = 0$, $(\alpha, \beta_i) = [\alpha, w_i]$,

(8) $\Sigma_i \dfrac{db_i}{d\alpha} [\alpha', w_i] = \Sigma_i \dfrac{db_i}{d\alpha'} [\alpha, w_i]$.

Also from (6),

$$\frac{\partial}{\partial \alpha} [\alpha', w] - \frac{\partial}{\partial \alpha'} [\alpha, w] = \frac{\partial}{\partial w} [\alpha', \alpha] = 0 \,;$$

hence

(9) $[\alpha_i, w_j] = (\alpha_i, \beta_j) = \dfrac{dc_j}{d\alpha_i} = \dfrac{\partial^2}{\partial \alpha_i \partial w_j} (c_1 w_1 + c_2 w_2 + c_3 w_3)$ $(i, j = 1, 2, 3)$,

where c_1, c_2, c_3 are functions of the arbitrary and known constants of the α-set only ; they are thus completely defined in accordance with the remark at the end of art. 2.

Equation (8) then gives

$$\Sigma_i \frac{db_i}{d\alpha} \frac{dc_i}{d\alpha'} = \Sigma_i \frac{db_i}{d\alpha'} \frac{dc_i}{d\alpha},$$

or

$$\frac{d}{d\alpha} \left(\Sigma_i b_i \frac{dc_i}{d\alpha'} \right) = \frac{d}{d\alpha'} \left(\Sigma_i b_i \frac{dc_i}{d\alpha} \right),$$

showing that

(10) $\Sigma_i b_i \, dc_i = - \, dB$,

a perfect differential.   This gives a complete definition of B if we assume that it contains no term independent of the arbitrary constants.

5. *Canonical systems.*—The constants α_i can in general be chosen arbitrarily. Take $\alpha_i = c_i$.   Then

$$[c_i, w_i] = (c_i, \beta_i) = 1, \qquad [c_i, w_j] = (c_i, \beta_j) = 0 \qquad (j \neq i).$$

The choice is possible if the Jacobian of the c_i with respect to the α_i is not zero ; in the applications to be made it is easy to see that this is the case.   Thus the c_i are assumed to be independent.   We then have from art. 4,

$$b_i = - \frac{dB}{dc_i},$$

where B is expressed in terms of the c_i and the known constants.

All these results are perfectly well known.   The sets c_i, β_i and c_i, w_i are canonical systems, for the (f, g), $[f, g]$ are nothing else than LAGRANGE's brackets and the above values reduce LAGRANGE's equations for the variation

of arbitrary constants, when a disturbing function R is added to F, to the canonical forms

$$\frac{dc_i}{dt} = \frac{dR}{d\beta_i}, \qquad \frac{d\beta_i}{dt} = -\frac{dR}{dc_i},$$

and

$$\frac{dc_i}{dt} = \frac{dR'}{dw_i}, \qquad \frac{dw_i}{dt} = -\frac{dR'}{dc_i},$$

where $R' = R + B$ and R is supposed to be expressed in terms of the w_i, c_i, t and known constants.

6. Another canonical set is obtained by putting

$$B_i = \frac{dc_1}{d\alpha_i}\beta_1 + \frac{dc_2}{d\alpha_i}\beta_2 + \frac{dc_3}{d\alpha_i}\beta_3.$$

and taking α_i, B_i as arbitraries; this is evident from equation (9). The same equation shows that if we put

$$W_i = \frac{dc_1}{d\alpha_i}w_1 + \frac{dc_2}{d\alpha_i}w_2 + \frac{dc_3}{d\alpha_i}w_3,$$

the α_i, W_i form a canonical set.

The sets c_i, β_i; α_i, B_i; α_i, W_i have the disadvantage that when we form the equations of variations the time appears as a factor of the periodic terms. This fact prevents them from being useful for most purposes of the lunar theory to which application will chiefly be made. It is to be noticed nevertheless that when β_i or w_i are taken as one set of arbitraries, the c_i is the conjugate set which makes the whole canonical; if α_i be taken as one set then the B_i or W_i has the same property with reference to the α_i. In general, if one set of constants be given the second set which renders the whole canonical is determined uniquely; the apparent duplicity in the above is due to the fact that W_i, w_i are not arbitrary *constants* but contain the time.

If it be possible to choose the b_i as one set of arbitraries, the conjugate sets are

$$\Sigma_j \beta_j \frac{dc_j}{db_i} = \Sigma_j \beta_j \frac{dc_i}{db_j}, \qquad \Sigma_j w_j \frac{dc_j}{db_i} = \Sigma_j w_j \frac{dc_i}{db_j} \qquad (j=1, 2, 3);$$

and from (10),

$$c_i = \frac{d}{db_i}(B + \Sigma_j b_j c_j),$$

so that

$$(b_i, \beta_j) = [b_i, w_j] = \frac{d^2}{db_i\, db_j}(B + b_1 c_1 + b_2 c_2 + b_3 c_3).$$

§ 3. APPLICATION TO THE KNOWN CONSTANTS.

7. Suppose that F contains a known constant α_0 of the α-set and a known angle $w_4 = b_4 t + \beta_4$ in which (the more general case) b_4 is a function of α_0 and β_4 belongs to the β set. Then applying (6) successively to the various combinations of α_0, w_4 with the arbitraries and using (7), (9) we obtain

$$\frac{\partial}{\partial \alpha'} [\alpha, w_4] - \frac{\partial}{\partial \alpha} [\alpha', w_4] = \frac{\partial}{\partial w_4} [\alpha, \alpha'] = 0,$$

$$\frac{\partial}{\partial \alpha'} [\alpha, \alpha_0] - \frac{\partial}{\partial \alpha} [\alpha', \alpha_0] = \frac{\partial}{\partial \alpha_0} [\alpha, \alpha'] = 0,$$

$$\frac{\partial}{\partial \alpha} [w, w_4] - \frac{\partial}{\partial w} [\alpha, w_4] = \frac{\partial}{\partial w_4} [w, \alpha] = 0,$$

$$\frac{\partial}{\partial \alpha} [\alpha_0, w] = \frac{\partial}{\partial \alpha_0} [\alpha, w] - \frac{\partial}{\partial w} [\alpha, \alpha_0],$$

$$\frac{\partial}{\partial \alpha} [\alpha_0, w_4] = \frac{\partial}{\partial \alpha_0} [\alpha, w_4] - \frac{\partial}{\partial w_4} [\alpha, \alpha_0].$$

The first three of these give

$$(11) \qquad [\alpha_i, w_4] = \frac{\partial C_4}{\partial \alpha_i}, \quad [\alpha_i, \alpha_0] = \frac{\partial C_0}{\partial \alpha_i}, \quad [w_i, w_4] = \frac{\partial C_4}{\partial w_i} \quad (i = 1, 2, 3),$$

and the last two then give

$$(12) \qquad [\alpha_0, w_i] = \frac{dc_i}{d\alpha_0} - \frac{\partial C_0}{\partial w_i}, \qquad [\alpha_0, w_4] = \frac{\partial C_4}{\partial \alpha_0} - \frac{\partial C_0}{\partial w_4} \quad (i = 1, 2, 3),$$

in which C_4, C_0 are, owing to the forms of the square brackets as deduced from the expressions for the coördinates, trigonometric series with the arguments w_i, w_4 and with coefficients depending on α_i, α_0; C_4 consists of cosines only, C_0 of sines only. Moreover, C_4 is only defined by its derivates with respect to α_0, α_i, w_i and may therefore contain terms independent of them, while C_0 is only defined by its derivatives with respect to α_i, w_i, w_4 and might contain terms independent of them. The suppositions of § 1 prevent the presence of such terms, so that C_0, C_4 are fully defined.

If the solution of the original equations has been obtained algebraically we have then sufficient material to find C_4, C_0. It may happen, however, that numerical values have been given to the constants in forming the solution, and for many purposes it is useful to possess equations for C_4, C_0 which will enable us to find their values in this case. It will be shown that they satisfy ordinary differential equations of the first order.

These equations may be obtained directly from the results of arts. 3, 4, 7 without using the original equations of motion. But as they involve $\partial' F/\partial \alpha_0$, $d' F/\partial w_4$, it is much easier to obtain them directly from the results of arts. 4, 7 and the equations of motion. This latter method, which also furnishes the vis viva equation, will be followed.

In this section, unless explicitly stated otherwise, the suffixes i, j receive only the values 1, 2, 3, that is, the letters to which they are attached are *arbitraries* of the solution.

8. *The Vis Viva equation.* Since

$$[\alpha_i, w_j] = \frac{dc_j}{d\alpha_i}, \qquad [\alpha_i, w_4] = \frac{dC_4}{d\alpha_i},$$

$$\frac{dQ}{dt} = \Sigma_i b_i \frac{dQ}{dw_i} + b_4 \frac{dQ}{dw_4},$$

where Q contains the time only through its presence in the angles w_i, w_4, we have

$$\Sigma_j b_j \frac{dc_j}{d\alpha_i} + b_4 \frac{dC_4}{d\alpha_i} = S \left(\frac{\partial \dot{x}}{\partial \alpha_i} \dot{x} - \ddot{x} \frac{\partial x}{\partial \alpha_i} \right) = \frac{\partial}{\partial \alpha_i} (T - F).$$

But $\Sigma_j b_j dc_j = - dB$. Therefore, since there are no terms independent of the arbitrary constants and b_4 is a known constant,

(13) $$T - F - b_4 C_4 = - B.$$

9. *The equation for C_4.* Differentiating (13) with respect to the time,

$$b_4 \frac{dC_4}{dt} = \frac{d}{dt} (T - F) = S (x \dot{x} - \ddot{x} \dot{x}) - b_4 \frac{d' F}{dw_4},$$

or

(14) $$\frac{dC_4}{dt} = - \frac{d' F}{dw_4}.$$

The variable C_4 is therefore similar to the L of M. POINCARÉ in his paper *Sur les équations de la lune.* * We might have defined C_4 by means of this equation and have then reversed the processes of this article and obtained the first and third of equations (11). As C_4 can contain a constant term, the definition is not complete, but will be made so by the equation obtained in art. 9.

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* Bulletin Astronomique, vol. 17 (1900), p. 167.

10. *The equation for C_0.*—Multiply the first of equations (12) by b_i, the second by b_4 and form the sum. We obtain

$$\Sigma_i b_i \frac{dc_i}{d\alpha_0} + b_4 \frac{\partial C_4}{\partial \alpha_0} - \frac{dC_0}{dt} = S\left(\frac{\partial \dot{x}}{\partial \alpha_0} x - \ddot{x}\frac{\partial x}{\partial \alpha_0}\right)$$

$$= \frac{\partial T}{\partial \alpha_0} - \frac{\partial F}{\partial \alpha_0} + \frac{\partial' F}{\partial \alpha_0}$$

$$= \frac{\partial}{\partial \alpha_0}(b_4 C_4) - \frac{dB}{d\alpha_0} + \frac{\partial' F}{\partial \alpha_0},$$

by (13). In this equation B is supposed to be expressed in terms of α_i, α_0. Let $dB_c/d\alpha_0$ denote the derivative of B with respect to α_0 when B is expressed in terms of c_1, c_2, c_3, α_0, so that

$$\frac{dB}{d\alpha_0} = \frac{dB_c}{d\alpha_0} + \Sigma_i \frac{dB_c}{dc_i}\frac{dc_i}{d\alpha_0} = \frac{dB_c}{d\alpha_0} - \Sigma_i b_i \frac{dc_i}{d\alpha_0}.$$

Substituting in the former equation and rearranging, we obtain

(15) $$\frac{dC_0}{dt} + C_4 \frac{db_4}{d\alpha_0} - \frac{dB_c}{d\alpha_0} = -\frac{\partial' F}{\partial \alpha_0},$$

the equation required.

Since C_0 contains no constant term, (15) gives it completely when C_4 has been obtained. We already know C_4 from (14) except its constant term. But since C_0 contains no term proportional to the time, the constant term in C_4 is given by

(15)⁰ $$\frac{db_4}{d\alpha_0}(C_4)^0 - \frac{dB_c}{d\alpha_0} = -\left(\frac{\partial' F}{\partial \alpha_0}\right)^0,$$

where $(\,\cdot\cdot\,)^0$ denotes the constant term in the expansion of the enclosed function in trigonometric series.

If b_4 does not depend on α_0, it is easily seen that the equation has its second term absent and C_0 can be found independently of C_4.

But now the constant part of C_4 cannot be determined by (15)⁰. The difficulty is turned by following the investigation of this article with b_4 as the constant instead of α_0. Then as b_4 occurs explicitly in F only through its presence in w_4 the equation (15) reduces to

$$\frac{dC_0}{dt} + C_4 - \frac{dB_c}{db_4} = 0,$$

giving

$$(C_4)^0 = \frac{dB_c}{db_4}.$$

The C_0 of this equation is, of course, different from that of (15).

There may be other known constants α_p present in F. If b_4 depends on α_p then there will be an equation for C_p corresponding to (15). If b_4 is independent of α_p, the equation satisfied by the corresponding C_p is

$$(16) \qquad \frac{dC_p}{dt} - \frac{dB_c}{d\alpha_p} = -\frac{\partial' F}{\partial \alpha_p},$$

where C_p consists of sines only. It will be convenient in what follows to suppose that any other constants are all of the latter type and therefore may either be treated as numerical constants in which case the corresponding equation (16) will be unnecessary, or as algebraic constants in which case it may be necessary to use the equation. Both cases may arise in the method of the variation of known constants which is developed below.

11. Suppose that F is homogeneous of order q with respect to x, y, z, and a certain constant A present in F, there being also present an absolute constant (e. g., the "gravitation" constant) which restores the proper physical dimensions of the equations of motion.

Then we have, symbolically,

$$F, B, T = q \text{ (length)}.$$

The equations of motion show that

$$\text{time} = (1 - \tfrac{1}{2}q) \text{ (length)},$$

and since $[c_i, w_i] = 1$, $b = -1$ (time),

$$c_i = (1 + \tfrac{1}{2}q) \text{ (length)} = \frac{q+2}{q-2}(b_i), \qquad F, B, T = \frac{2q}{q+2}(c_i).$$

The choice of α_0 has been quite arbitrary. When it is a function of b_4 take it so that it is of the same dimensions as the c_i and further such that

$$\frac{d\alpha_0}{db_4} = \frac{q+2}{q-2}\frac{\alpha_0}{b_4}.$$

As A is merely defined as a length, choose it so that

$$\frac{dA}{A} = \frac{2}{q+2}\frac{d\alpha_0}{\alpha_0}.$$

Thus A, α_0 are defined in terms of b_4, but the manner in which they occur in F is still arbitrary except with respect to their dimensions.

Since B will be homogeneous with respect to the c_i, α_0 we have

$$\alpha_0 \frac{dB_c}{d\alpha_0} = \frac{2q}{q+2} B - \Sigma_i c_i \frac{dB}{dc_i} = \frac{2q}{q+2} B + \Sigma_i b_i c.$$

12. Multiply (15) by $q + 2$ and apply the results for dA, $d\alpha_0$ in art. 11. We obtain

$$- (q + 2)\alpha_0 \frac{dC_0}{dt} - (q - 2)b_4 C_4 + (q + 2)\alpha_0 \frac{dB_c}{d\alpha_0} = 2A\,\frac{\partial' F}{\partial A},$$

where F is supposed to be expressed in terms of A instead of α_0.

Now F is homogeneous of order q with respect to x, y, z, A. Hence, multiplying equations (1) by $2x$, $2y$, $2z$, and adding their sum to the last equation, we obtain

$$\frac{d^2}{dt^2} r^2 - 4T - (q + 2)\alpha_0 \frac{dC_0}{dt} - (q - 2)b_4 C_4 + (q + 2)\alpha_0 \frac{dB_c}{d\alpha_0} = 2qF,$$

which, by (15) and the last result in art. 11, becomes

$$\frac{d^2}{dt^2} r^2 - (q + 2)\alpha_0 \frac{dC_0}{dt} - (q - 2)(T - F + B) + 2qB$$
$$+ (q + 2)\Sigma b_i c_i = 2qF + 4T,$$

or, reducing,

$$(17) \qquad T + F - B - b_1 c_1 - b_2 c_2 - b_3 c_3 = \frac{d}{dt}\left(\frac{1}{2 + q} \frac{d}{dt} r^2 - \alpha_0 C_0' \right).$$

Denote by zero index the constant terms in the expansion of these functions. Then, since the right-hand member contains no constant term,

$$(18) \qquad\qquad T^0 + F^0 = B + b_1 c_1 + b_2 c_2 + b_3 c_3,$$

a result independent of q.

13. The equation (18) gives a useful formula for finding the c_i when the solution has been obtained with any set of arbitraries and even when numerical values have been given to the α-set. For

$$c_i = \frac{\partial}{\partial b_i}(B + \Sigma b_i c_i) = \frac{\partial}{\partial b_i}(T^0 + F^0)$$

$$= S\left[\dot{x}\frac{\partial \dot{x}}{\partial b_i} + \ddot{x}\frac{\partial x}{\partial b_i} \right]^0 = S\left[\dot{x}\frac{dx}{dw_i} + \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial b_i} \right) \right]^0.$$

or

$$(19) \qquad\qquad c_i = \left[\dot{x}\frac{\partial x}{\partial w_i} + \dot{y}\frac{\partial y}{\partial w_i} + \dot{z}\frac{\partial z}{\partial w_i} \right]^0 \qquad\qquad (i = 1, 2, 3).$$

Define a new constant c_4 by a similar equation for $i = 4$. We then have

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 = 2T^0,$$

or by (18),

$$T^0 - F^0 - b_4 c_4 = - B.$$

Comparing with the vis viva equation, we obtain

$$(20) \qquad\qquad c_4 = (C_4)^0.$$

§ 4. CANONICAL EQUATIONS OF MOTION.

14. The equations (14), (15) may be written

$$(21) \qquad \frac{dC_4}{dt} = - \frac{\partial' F}{\partial w_4}, \qquad \frac{d}{dt}\left(C_0 - t\frac{\partial B_c}{\partial \alpha_0} \right) = - \frac{\partial' F}{\partial \alpha_0} - C_4 \frac{db_4}{d\alpha_0}.$$

These with the equations of motion (1) suggest the following transformation. Put

$$2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \qquad W_0 = - C_0 - b_0 t + \beta_0, \qquad \phi = T - F - b_4 C_4,$$

where β_0 is a constant. If we consider x, y, x, C_4, α_0 and $\dot{x}$, $\dot{y}$, $\dot{z}$, w_4, W_0 as independent variables and ϕ expressed as a function of them, the equations (1) and (21) may be written in the canonical form

$$(22) \qquad \left\{ \begin{array}{ll} \dfrac{dx}{dt} = \dfrac{\partial \phi}{\partial \dot{x}}, & \dfrac{d\dot{x}}{dt} = - \dfrac{\partial \phi}{\partial x}; \quad \cdots, \quad \cdots; \quad \cdots, \quad \cdots; \\[3mm] \dfrac{dC_4}{dt} = \dfrac{\partial \phi}{\partial w_4}, & \dfrac{dw_4}{dt} = - \dfrac{\partial \phi}{\partial C_4}; \end{array} \right.$$

$$(23) \qquad \frac{d\alpha_0}{dt} = \frac{\partial \phi}{\partial W_0}, \qquad \frac{dW_0}{dt} = - \frac{\partial \phi}{\partial \alpha_0};$$

for ϕ does not contain W_0 explicitly, α_0 is a constant, and $dw_4/dt = b_4$. The same equations hold if b_4 is not a function of α_0.

We may either use the first four or all five pairs of equations and the solutions of such canonical systems are well known. The last pair gives $\alpha_0 =$ constant and defines W_0. It will be more convenient, however, to treat α_0 as a given constant and use only the first four equations in the following work.

When (22) are solved canonically in terms of the constants c_j and the angles w_j ($j = 1, 2, 3, 4$) we find

$$\phi = \text{constant} = - B', \qquad b_j = - \frac{dB'}{dc_j}.$$

The previous articles show that $B' = B + b_4 c_4$ where B is independent of c_4 [the assumption that the constant parts of ϕ, etc., contain no parts independent of c_1, c_2, c_3 no longer holds]. Hence, as before, $b_i = - \partial B_4/\partial c_i$. Also the expressions for the coördinates are independent of c_4.

15. All the preceding investigations might have been developed by starting with these canonical equations and solving them by means of JACOBI's method, obtaining the results in the inverse order. I have preferred to follow the more natural if somewhat longer development as it seems to make the deduction of some of the results rather more clear in view of the applications which follow.

If there be angles other than w_4 and constants other than α_0 present in F, similar results may be obtained. In fact, the extension of this method to any number of variables, angles and constants will be sufficiently obvious. To each angle and constant corresponds respectively a pair of canonical equations; to ϕ must be added one term for each angle present. When this has been done, ϕ, considered as a function of all the variables, is independent of t explicitly and $\phi = $ const. is an integral of the equations.

§ 5. VARIATION OF CONSTANTS.

16. *Variation of arbitrary constants.*—Let a new function ψ be added to ϕ, then, using the canonical set c_1, c_2, c_3, c_4, w_1, w_2, w_3, w_4, the equations satisfied by them are

$$\frac{dc_j}{dt} = -\frac{\partial}{\partial w_j}(\psi - B') = -\frac{\partial \psi}{\partial w_j}, \qquad \frac{dw_j}{dt} = \frac{\partial}{\partial c_j}(\psi - B') \quad (j = 1, 2, 3, 4),$$

where ψ and B' are expressed in terms of c_j, w_j, of the given constants and the time, B' being a function of the c_j and the constants present in ϕ.

The method appears to fail if ψ contains none of the arbitraries c_j, w_j. But the equations of motion depend on the *derivatives* of ψ with respect to the variables; we can therefore assume that ψ is not independent of the arbitraries.

If ψ is independent of the w_j, the c_i are constant and the equations for w_j are easily solved. If ψ contains a small factor α and a term with argument $bt + \beta$, where b is small and β independent of the β_i, the corresponding terms in w_i will contain divisors b, b^2/α, b^3/α^2; $\cdots$; when β is not independent of β_j the first divisor will in general be b^2.

17. *Variation of the given constants.* Let ψ be due to a small addition $\delta\alpha_0$ to α_0, one of the constants present in F where $\delta\alpha_0$ may be a constant or a function of the time. Then using the canonical equations (22) and assuming that $\delta\alpha_0$ produces variations δc_j, δw_j and that squares of the variations may be neglected,

$$\psi = \Sigma_j\left(\frac{\partial \phi}{\partial c_j}\delta c_j + \frac{\partial \phi}{\partial w_j}\delta w_j + \frac{\partial \phi}{\partial \alpha_0}\delta \alpha_0\right) = -\Sigma_j\frac{dB'}{dc_j}\delta c_j - \frac{dB'}{d\alpha_0}\delta\alpha_0,$$

since B' is independent of the w_j.

The equations of variations are

$$\frac{d}{dt}\delta c_j = -\frac{\partial \psi}{\partial w_j} = \Sigma_{j'}\frac{dB'}{dc_{j'}}\frac{\partial}{\partial w_j}\delta c_{j'} + \frac{dB'}{d\alpha_0}\frac{\partial}{\partial w_j}\delta\alpha_0,$$

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{\partial \psi}{\partial c_j} + b_j = -\frac{\partial}{\partial c_j}\left(\Sigma_{j'}\frac{dB'}{dc_{j'}}\delta c_{j'} + \frac{dB'}{d\alpha_0}\delta\alpha_0\right) + b_j$$

$$(j, j' = 1, 2, 3, 4).$$

As squares of the variations are neglected we can substitute constant values for the coefficients of the variations in the right hand members and therefore

$$\frac{d}{dt}\delta c_j = -\Sigma_{j'}b_{j'}\frac{\partial}{\partial w_j}\delta c_{j'} + \frac{dB'}{d\alpha_0}\frac{\partial}{\partial w_j}\delta \alpha_0,$$

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{\partial}{\partial c_j}\left(\Sigma_{j'}b_{j'}\delta c_{j'} - \frac{dB'}{d\alpha_0}\delta \alpha_0\right) + b_j.$$

18. Suppose that $\delta \alpha_0$ is independent of the w_j, c_j. The last terms of the equations for δc_j then vanish and they are satisfied by

$$\delta c_j = 0.$$

Substituting in the second set we find, since $B' = B + b_4 c_4$,

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{db_j}{d\alpha_0}\delta \alpha_0 + b_j.$$

That is, in order to obtain the new value of w_j, all that is necessary is to substitute $\alpha_0 + \delta \alpha_0$ for α_0 in the expression for b_j expressed as a function of c_1, c_2, c_3, α_0.

Suppose that the original solution had been expressed in terms of any arbitraries α_1, α_2, α_3 of the α-set, and of the known constant α_0; the c_i having been obtained in terms of them by the method of art. 13. Then the variations of α_i are found by solving the equations

$$\frac{dc_i}{d\alpha_1}\delta \alpha_1 + \frac{dc_i}{d\alpha_2}\delta \alpha_2 + \frac{dc_i}{d\alpha_3}\delta \alpha_3 = -\frac{dc_i}{d\alpha_0}\delta \alpha_0 \qquad (i=1,2,3),$$

for $\delta \alpha_1$, $\delta \alpha_2$, $\delta \alpha_3$. The changes in the coefficients of the periodic terms are obtained by substituting these changes of the constants therein. But the changes in the b_i are given by

$$\delta b_i = \frac{db_i}{d\alpha_1}\delta \alpha_1 + \frac{db_i}{d\alpha_2}\delta \alpha_2 + \frac{db_i}{d\alpha_3}\delta \alpha_3 + \frac{db_i}{d\alpha_0}\delta \alpha_0 \qquad (i=1,2,3),$$

that is to say, we must also substitute the change of α_0 as well as those of α_1, α_1, α_3 in the b_i.

The same results hold if b_4 is not a function of α_0.

19. If ψ be due to a variation δw_4 of w_4, we have in a similar manner,

$$\psi = \Sigma_j b_j \delta c_j;$$

and the equations of variations are

$$\frac{d}{dt}\delta c_j = -\Sigma_{j'}b_{j'}\frac{\partial}{\partial w_j}\delta c_{j'}, \qquad \frac{d}{dt}(w_j + \delta w_j) = -\frac{\partial}{\partial c_j}\left(\Sigma_j\frac{dB'}{dc_{j'}}\delta c_{j'}\right) + b_j.$$

Suppose that the variation of w_4 is given. The left hand member of the last of the second set, namely,

$$\frac{d}{dt}\delta w_4 = \Sigma_{j'} b_{j'} \frac{\partial}{\partial c_4} \delta c_{j'}$$

is known.

The value of δw_4 may consist of periodic and non-periodic terms: suppose it to be independent of the angles w_1, w_2, w_3 and of c_1, c_2, c_3. Let us try if $\delta c_1 = 0$, $\delta c_2 = 0$, $\delta c_3 = 0$ and δc_4 independent of w_1, w_2, w_3 can satisfy the equations. The first three of the first pair are satisfied and the others become

$$\frac{d}{dt}\delta c_4 = -b_4 \frac{\partial}{\partial w_4}\delta c_4 \quad \frac{d}{dt}\delta w_4 = b_4 \frac{\partial}{\partial c_4}\delta c_4, \quad \frac{d}{dt}\delta w_i = b_4 \frac{\partial}{\partial c_i}\delta c_4 \quad (i=1,2,3).$$

If $\delta w_4 = qt + q'$, where q, q' are given constants, the equations are satisfied by $\delta c_4 = qc_4/b_4$, $\delta w_i = 0$. It does not appear to be possible in general to satisfy the equations with other values for δw_4.

§ 6. APPLICATIONS TO THE LUNAR THEORY.

20. The results contained in the previous sections are more general in their character than is necessary for the applications to most problems in celestial mechanics and in particular for that one which is known as the lunar theory. Without going into details it is sufficient to state here that the few restrictions imposed are satisfied in these applications.

Let x, y, z, r denote the coördinates and distance of the moon relatively to the earth's center; x', y', z', r', those of the sun relatively to the center of mass of the earth and moon; $\mu = E + M$, the sum of the masses of the earth and moon; m' the mass of the sun; let the three bodies be considered to attract in the same way as particles of the same masses situated at their centers of mass respectively. Also suppose that the motion of the sun is known and that all other influences are neglected. Then it is well known that

$$F = \frac{\mu}{r} + m'\{r^2 - 2r(xx' + yy' + zz') + r'^2\}^{-\frac{1}{2}} - m'(xx' + yy' + zz')r'^{-3},$$

where, after the substitution of polar coördinates for x', y', z' and expansion in powers of $1/r'$, the term $1/r'^3$ is dropped and the coefficient of $(r')^{-p}$ is multiplied by

$$\kappa_p = \left(\frac{E}{E+M}\right)^{p-1} - \left(\frac{-M}{E+M}\right)^{p-1}.$$

We can therefore put

$$F = \frac{\mu}{r} + m'\sum_{p=2}^{\infty} \kappa_p R'_p (r')^{-p-1},$$

where R'_p is a homogeneous function of order p with respect to x, y, z, and also

contains trigonometric functions of the angular position of the sun.   Further, if n', a' be the mean motion and mean distance of the sun, $m' + \mu = n'^2 a'^3$, or, putting $\lambda = \mu/(m' + \mu)$,

$$m' = n'^2 a'^3 (1 - \lambda).$$

We shall now suppose that the plane of the sun's orbit is fixed and that its motion can always be expressed in terms of n', e' and the angle $n't + \beta_4$; when its motion is elliptic n', β_4, e' are constants.   Properly, there are two angles $n't + \epsilon'$, $n't + \epsilon' - \pi'$, but well known results show that we may treat them as though π' were zero and they also indicate where π' has to be inserted in the final results.   Thus we can put

$$F = \frac{\mu}{r} + n'^2 (1 - \lambda) \sum_{p=2}^{\infty} \frac{R_p}{a'^{p-2}},$$

where R_p is homogeneous and of order p relatively to x, y, z and contains κ_p, e' and trigonometric functions of $n't + \epsilon'$.

21. We have from the equations of motion,

$$\frac{1}{2} \frac{d^2}{dt^2} r^2 - 2T = Sx \frac{\partial F}{\partial x} = -\frac{\mu}{r} + n'^2 (1 - \lambda) \Sigma_p p R_p (a')^{-p+2} \quad (p=2, 3, \cdots).$$

Eliminating T between this equation and (17) we find, since $q = -1$ here,

$$(24) \quad \frac{3}{2} \frac{\mu}{r} - \frac{1}{2} n'^2 (1-\lambda) \Sigma_p (p-2) R_p (a')^{-p+2} - B - \Sigma_i b_i c_i = \frac{3}{4} \frac{d^2}{dt^2} r^2 - \alpha_0 \frac{dC_0}{dt}.$$

The right hand member contains no constant term.   Hence

$$(25) \quad B + b_1 c_1 + b_2 c_2 + b_3 c_3 = \frac{3}{2} \left(\frac{\mu}{r}\right)^0 - \frac{1}{2} n'^2 (1 - \lambda) \Sigma_p (p - 2)(R_p)^0 (a')^{-p+2}.$$

The equations (14), (15) show that dC_0/dt is a quantity whose lowest order is n'^2/α_0.   As it is usual in the lunar theory to expand in powers of n' and $1/a'$, it will be more convenient to put $\alpha_0 C_0 = C_0'$ so that dC_0'/dt is a quantity whose lowest order is n'^2 and is expansible like the coördinates.   We therefore have

$$(26) \quad \frac{dC_0'}{dt} = \frac{3}{4} \frac{d^2}{dt^2}(r^2) - \frac{3}{2} \frac{\mu}{r} + \frac{1}{2} n'^2 (1 - \lambda) \Sigma_p (p - 2) R_p (a')^{-p+2} + f$$
$$(p = 3, 4, \cdots),$$

f being a constant so chosen as to make the constant part of the right-hand member zero.

22. The properties of equations (25), (26) which are remarkable and valuable for practical purposes arise principally in those investigations which permit us to neglect the ratio of the parallaxes of the sun and moon.   This ratio is

about $\frac{1}{400}$ and for most of the purposes which require the value of C'_0, i. e., in the calculation of the indirect action of a planet (art. 24 below), it is possible to neglect it.   But of the R_p, the first term R_2 is the only one which is independent of it and the factor $p - 2$ causes R_2 to disappear in both the equations. The equation (26) then reduces to

$$(27) \qquad\qquad \frac{dC'_0}{dt} = \tfrac{3}{4} \frac{d^2}{dt^2}(r^2) - \tfrac{3}{2}\left(\frac{\mu}{r}\right) + \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0.$$

Another observation may be made.   If the object in view is such that we only require those terms in C'_0 which are of long period relatively to the month, i. e., those in which the coefficients of t in the arguments are small, the first term of the right-hand member will give rise to coefficients which are very small compared with those arising from the second member; in such cases we can generally put

$$\frac{dC'_0}{dt} = - \tfrac{3}{2}\left(\frac{\mu}{r}\right) + \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0,$$

without committing sensible errors.

23. In nearly all cases of the indirect action of a planet, we can neglect the *square* of the ratio of the parallaxes.   But it is well known that the constant parts of the functions we have been dealing with are expansible in powers of the square of this ratio.   *Hence neglecting the square of the ratio of the parallaxes,*

$$B + b_1 c_1 + b_2 c_2 + b_3 c_3 = \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0.$$

This is the result which I obtained in an earlier paper * and which I used to prove ADAMS' well known theorems and to obtain the secular accelerations directly from the expression for the parallax of the moon.   It is true that the square of the ratio of the parallaxes was neglected, but it was easy to see from the few terms which DELAUNAY gave in a literal form that they could exert no sensible effect on the result.   If doubts might still arise, owing to slow convergence, it is not difficult to remove them by means of the numerical results of my lunar theory taken in conjunction with equation (25).

The methods used here differ in form from those which were adopted in the paper just referred to.   The essential feature of the earlier method was really that of DELAUNAY, that is to say, the elliptic orbit was used as a first approximation and the variation of arbitrary constants applied to it somewhat in DELAUNAY's manner.   Here we are theoretically quite independent of the elliptic orbit, and any set of arbitrary constants which possesses the property of enabling

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* See footnote, p. 334 above.

us to expand in trigonometric series may be used.   This has the practical advantage of permitting the calculation of c_1, c_2, c_3, by means of art. 13 from the results of any theory in whatever form it may be expressed and whatever be the arbitrary constants used.   Provisionally it may be best to except the theory of HANSEN, although it is probable that the expression $\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z$, which is the principal part of equation (19), might be transformed so as to be used in this case also without serious difficulty.

24. The results of arts. 15, 16 include NEWCOMB's theorem concerning the secular accelerations and the indirect action of a planet * as a particular case. In this valuable memoir he considered the action of a planet as a case of the disturbed motion of three bodies, that is to say the problem of the relative motion of the earth, moon and sun is considered to have been solved and the question was then to find out what variable values the 12 arbitrary constants which arise in the solution must have in order that the solution may retain the same form when another disturbing cause acts.   He showed that if we first consider the orbit of the sun as elliptic and if the action of a planet causes variations $\delta e'$, $\delta n'$ in e', n', then the new non-periodic parts of n, π_1, θ_1 (the mean motions of the moon, its perigee and node) are found by solving the equations

$$\frac{\partial c_i}{\partial n}\,\delta n + \frac{\partial c_i}{\partial e}\,\delta e + \frac{\partial c_i}{\partial \gamma}\,\delta\gamma + \frac{\partial c_i}{\partial n'}\,\delta n' + \frac{\partial c_i}{\partial e'}\,\delta e' = 0 \qquad (i = 1, 2, 3)$$

for δn, δe, $\delta\gamma$, regarding $\delta n'$, $\delta e'$ as known, and substituting the variable values of n, e, γ, n', e' in the expressions of n, π_1, θ_1 in terms of these five quantities. He remarks, "This curious theorem may embody some principle applicable to the disturbed motion of three bodies, which has not yet been fully mastered."

It appears from the investigation of this paper that the theorem is independent of the particular form in which the solar coördinates enter into the lunar disturbing function and further, that it is true for any of the constants which may be present in this disturbing function, provided always that the given variations of the known constants are independent of the lunar elements.   Finally, it is applicable to any dynamical problem in which the coördinates satisfy the specified conditions and to any constants which may enter into the force-function in the manner specified at the outset.

HAVERFORD COLLEGE,
        April 20, 1903.

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* *Action of the Planets on the Moon*, Washington Astronomical Papers, vol. 5 (1894), p. 191.